where s is an abscissa measured along the body and n is an ordinate measured away from the body along the normal. The singular line then is given by$^{1-5}$

$$v^2 = [(\gamma - 1)/(\gamma + 1)](1 - u^2) \tag{1}$$

where v and u are components of velocity in the s and n direction, respectively, referred to the limiting speed. The choice of the s,n system renders Eq. (1) free from explicit dependence on the shape of the body.

As the partial differential equations governing the flow are integrated in the n direction across the shock layer, there results an approximating system of ordinary differential equations in a single variable s; this system then is integrated numerically. The ordinary derivatives of the unknowns at any value of s are expressed in terms of the data along the ordinate s = const. The ordinate, therefore, plays the role of a data line for the continuation of the solution in direction of increasing s.

The significance of relation (1) can be revealed as follows. The local speed of sound, nondimensionalized with respect to the limiting speed, is

$$a = \{[(\gamma - 1)/2](1 - u^2 - v^2)\}^{1/2}$$

Equation (1) may be written as

$$\{1 + [(\gamma - 1)/2]\}v^2 = [(\gamma - 1)/2](1 - u^2)$$
$$v^2 = [(\gamma - 1)/2](1 - u^2 - v^2) = a^2$$

Therefore, along the singular line the s component of velocity attains sonic value. Taking square roots and dividing by the magnitude of velocity q, one obtains

$$v/q = \pm a/q = \pm 1/M = \sin(\pm\mu) \tag{2}$$

where μ is the Mach angle. Equation (2) implies that the data line, s = const, becomes tangent to a characteristic at every point of the singular line along which Eq. (1) holds. In other words, the singular line is a locus of points at which the direction of integration becomes normal to a characteristic. Obviously, if the coordinate system is changed, so is the direction of integration, and the singularity moves to a different location. Incidentally, in spherical coordinates (r,θ), Eq. (1) has the same form if u and v are taken to be velocity components in r and θ directions.

Using the present interpretation, it is now possible to explain the well-known fact that the point of intersection of the singular line with the shock wave is nonsingular. This is so because the angle between the shock and the velocity immediately behind the shock is less than Mach angle. Consequently, the shock layer downstream of the singular line lies outside the range of influence of the point of intersection. This is borne out by the fact that the equation for the derivative of the shock angle, as given by integral method, is nonsingular.[4,5]

The implications of identifying the singular line of the integral method with the locus of tangency of data lines with characteristics are obvious in the light of the theory of characteristics. However, it should be pointed out that the occurrence of a singular line of the type described here is not a peculiarity of the method of integral relations. The method introduces no artificiality in the form of a singular line; on the contrary, it is rather remarkable to observe a complete agerement between an approximate method such as Bielotserkovskii's and the theory of characteristics.

References

[1] Bielotserkovskii, O. M., "Flow past a symmetric profile with detached shock wave," Prikl. Mat. i Mekh. (Appl. Math. Mech.) 22 (1958).

[2] Bielotserkovskii, O. M., "On the calculation of flow past axisymmetric bodies with detached shock waves using an electronic computing machine," Prikl. Mat. i. Mekh. (Appl. Math. Mech.) **24,** 511–517 (1960).

[3] Bielotserkovskii, O. M., "Calculation of flow past axisymmetric bodies with detached shock waves," *Equations for Calculations and Tables of Flow Fields* (Vichislitielnii Centr Akademii Nauk SSSR, Moscow, 1961).

[4] Traugott, S. C., "An approximate solution of the supersonic blunt body problem for prescribed arbitrary axisymmetric shapes," Research Rept. RR-13, Martin Co., Baltimore, Md. (August 29, 1858).

[5] Gold, R. and Holt, M., "Calculation of supersonic flow past a flat-head cylinder by Bielotserkovskii's method," Brown Univ., Div. Appl. Math., AFOSR TN 59-199 (March 1959).

Simultaneous Gas-Phase and Surface Atom Recombination for Stagnation Boundary Layer

PAUL M. CHUNG* AND SHAOWEN W. LIU†
Aerospace Corporation, El Segundo, Calif.

Nomenclature

C = $\rho\mu$ ratio defined by Eq. (3)
c = total atom mass fraction
c_2 = atom mass fraction of nitrogen atoms
D = binary diffusion coefficient
h_f = frozen total enthalpy
Δh^0 = heat of recombination
K_g = equivalent surface reaction constant for gas-phase reaction
$\hat{K}_g$ = parameter defined by Eq. (5)
K_w = specific catalytic surface recombination constant
k_0 = constant portion of recombination coefficient, 1.56×10^{20}
m = c/c_e
p = pressure, atm
q = heat transfer to surface
R = universal gas constant, 82.06 cm³-atm/mole-°K
r = distance from the axis of symmetry to the surface
s = function defined by Eq. (3)
Sc = Schmidt number, $\mu/\rho D$
T = absolute temperature, °K
u = streamwise velocity
x = streamwise distance
y = distance normal to surface
β = $(du_e/dx)_0$, sec for Eq. (8)
Γ_g = Damkhöler number for gas-phase recombination
Γ_w = Damkhöler number for surface recombination
ϵ = $\begin{cases} 0 \text{ for two-dimensional body} \\ 1 \text{ for axisymmetric body} \end{cases}$
η = function defined by Eq. (3)
μ = viscosity
ν = kinematic viscosity, μ/ρ
ρ = density

Subscripts

E = equilibrium
e = edge of boundary layer
f = frozen
0 = stagnation point
w = wall
∞ = freestream

Introduction

THE chemical state of nonequilibrium boundary layers about hypersonic vehicles is of considerable interest. The

---

Received by ARS December 7, 1962.

* Head, Heat Transfer Section, Aerodynamics and Propulsion Research Laboratory.

† Staff Scientist, Aerodynamics and Propulsion Research Laboratory.

interest stems mainly from the fact that the heat transfer to the body and the observables in the wake and the trails are affected greatly by the chemical state of the boundary layers.

The problem of nonequilibrium, chemically recombining boundary layers is a formidable one, and exact solutions are available only for a few special cases. The solutions that exist have been obtained by assuming that either the gas-phase or the surface recombination occurs alone.[1–6] At this stage of development, an approximate solution will be useful if it takes reasonable account of the simultaneous effects of gas-phase and surface recombination.

It has been shown[7] that the chemical state of the boundary layer is controlled mainly by the recombination that takes place near the wall when the wall is highly cooled. This observation suggests representing the gas-phase reaction by an equivalent surface reaction with all state variables specified by surface conditions. The order of reaction and the corresponding rate constant of this equivalent surface reaction will be determined for a noncatalytic wall. The equivalent surface reaction concept then will be extended to the case in which the surface has an arbitrary catalycity.

The basic concept of emphasizing the region of maximum chemical activity in determining the chemical state of the boundary layer first was used by Rae[8] for the boundary layer in which dissociative reactions predominate.

Analysis

Fay and Riddell[1] have obtained an exact solution of the nonequilibrium stagnation point boundary layer with the chemical state of the gas in the boundary layer ranging from frozen to equilibrium. The computations were for a noncatalytic wall at a temperature of 300°K and for a selected flight condition. Chung and Anderson[2] have calculated by an integral method the nonequilibrium effect on heat transfer to blunt cones with noncatalytic walls at temperatures of 1000° and 1500°K for two flight conditions. Goodwin and Chung[9] successfully have obtained additional solutions and have correlated the results for various flight conditions and a wall temperature of 1500°K by a single recombination rate parameter.

Subsequently, Inger[7] has shown that the difference in wall temperature between the two theoretical solutions[1, 2, 9] may be included in the correlation by noting the controlling influence of the wall temperature upon the chemical state of the boundary layer. It appears that, with the resulting comprehensive correlation, the essential effects of nonequilibrium reaction on chemical state and heat transfer have been taken into account, at least to the correct order of magnitude.

The order and rate constant for the proposed equivalent surface reaction will be determined first for noncatalytic walls. Since the gas-phase reaction near the surface of a highly cooled hypersonic vehicle is predominantly that of recombination requiring three body collisions, a modified second-order reaction appears to be a reasonable choice for the equivalent surface reaction. Let

$$\rho_w D_w(\partial m/\partial y)_w = K_g[m_w^2/(1 + m_w)] \tag{1}$$

where K_g is the equivalent surface reaction constant for the gas-phase reaction. The constant K_g is to be determined subsequently. In effect, the authors are assuming that the nonequilibrium gas-phase reaction over a noncatalytic wall is equivalent to a frozen boundary layer with the surface reaction specified by Eq. (1).

An exact solution of the diffusion equation at the stagnation point[3] gives the following equation for a frozen boundary layer:

$$(\partial m/\partial \eta)_w = 0.47(1 - m_w)Sc^{1/3} \tag{2}$$

where

$$\eta = \frac{u_e r^\epsilon}{(2Cs)^{1/2}} \int_0^y \rho dy$$

$$s = \int_0^x \rho_e \mu_e u_e r^{2\epsilon} dx$$

$$C = (\rho\mu/\rho_e\mu_e)_0 = \text{const} \tag{3}$$

Solution of Eqs. (1) and (2) for m_w gives‡

$$m_w = [1/(1 + \hat{K}_g)]^{1/2} \tag{4}$$

where

$$\hat{K}_g = \frac{Sc^{2/3}}{(0.47)\,[(1 + \epsilon)(\rho_e\mu_e)_0 \beta C]^{1/2}} K_g \tag{5}$$

For the Lewis number not much different from one, it readily can be shown that

$$(q - q_f)/(q_E - q_f) = 1 - m_w \tag{6}$$

Equations (4) and (6) give at the stagnation point

$$(q - q_f)/(q_E - q_f) = 1 - [1/(1 + \hat{K}_g)]^{1/2} \tag{7}$$

Solutions of nonequilibrium boundary layers exist for axisymmetric bodies only. The following correlations, therefore, will be limited to the stagnation regions of axisymmetric bodies. The correlation of Goodwin and Chung,[9] including the surface temperature correlation of Inger,[7] gives $(q - g_f)/(q_E - q_f)$ in terms of a single parameter:

$$\Gamma_g = \frac{1}{\beta}\,(2k_0)\,\frac{1}{(T_{e0})^{3.5}}\left(\frac{p}{R}\right)^2 \left[\frac{\Delta h^0}{h_{fe}}\,\frac{c_e^2}{(1 + c_e)}\,\frac{1}{(1 - c_{2e}^2)}\right] \times \left(\frac{1500}{T_w}\right)^{3.5} \tag{8}$$

It is desired to find $\hat{K}_g$ such that the heat transfer ratio $(q - q_f)/(q_E - q_f)$ obtained by Eq. (7) will match the result given by Goodwin and Chung in terms of Γ_g. It is found that the heat transfer ratios match quite satisfactorily if

$$\hat{K}_g = 21\Gamma_g \tag{9}$$

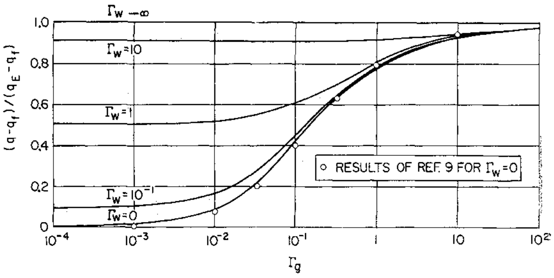

Fig. 1 Variation of $(q - q_f)/(q_E - q_f)$ with respect to Γ_w and Γ_g

Figure 1 shows the comparison between the heat transfer ratio obtained from Ref. 9 and that from Eqs. (7) and (9). It is seen that the present equivalent surface reaction theory represented by Eqs. (1, 7, and 9) predicts the heat transfer ratio satisfactorily for noncatalytic walls.

Consider now surfaces of arbitrary catalycity, represented by a specific rate constant K_w, in the presence of nonequilibrium gas phase reaction. The diffusion rate of atoms to the surface then must be equal to the total rate of recombination of atoms by both the gas-phase recombination and the catalytic surface recombination. Equation (1) is generalized for this case as

$$\rho_w D_w(\partial m/\partial y)_w = K_g[m_w^2/(1 + m_w)] + (\rho_w K_w)m_w \tag{10}$$

The solution of Eqs. (2, 10, and 6) for $(q - q_f)/(q_E - q_f)$ with

---

‡ For the stagnation region, $u_e = \beta x$ and $r = x$.

the aid of Eqs. (5) and (9) gives

$$\frac{q - q_f}{q_E - q_f} = 1 - \frac{[(\Gamma_w)^2 + 4(1 + \Gamma_w + 21\,\Gamma_g)]^{1/2} - \Gamma_w}{2(1 + \Gamma_w + 21\Gamma_g)} \tag{11}$$

where, for the stagnation region of axisymmetric bodies,

$$\Gamma_w = \frac{(\rho_w K_w)Sc^{2/3}}{(0.47)[2(\rho_e\mu_e)_0\beta C]^{1/2}} \tag{12}$$

Equation (11) is a closed form solution that gives the heat transfer ratio $(q - q_f)/(q_E - q_f)$ for arbitrary-rate simultaneous gas-phase and surface reactions.

The terms Γ_g and Γ_w, defined by Eqs. (8) and (12), are essentially Damkhöler numbers representing the ratio of characteristic diffusion time to characteristic chemical reaction time for the gas-phase and surface reactions, respectively.

Discussion

The accuracy of the solution, Eq. (11), cannot be checked for the arbitrary combinations of Γ_g and Γ_w because the general exact solutions are not available. It can be established that the equation gives correct results at the two limiting cases of either $\Gamma_g = 0$ or $\Gamma_w = 0$. It is sufficiently accurate when $\Gamma_w = 0$, because Γ_g was obtained by matching the results with the solutions of Ref. 9. It is correct also when $\Gamma_g = 0$, because the solution (11) essentially becomes the solution of Ref. 3. Though there is no direct check in the intermediate region, it is felt that the physical reasoning leading to the equivalent surface reaction concept of Eq. (10) is sound. The excellent correlation shown for the gas-phase reaction with noncatalytic surface appears to support the accuracy of the results presented.

The variation of $(q - q_f)/(q_E - q_f)$ with respect to Γ_g and Γ_w, as given by Eq. (11), is seen in Fig. 1. The general behavior is similar to that for the Couette flow obtained in Ref. 10.

It is noted here that the same problem considered here also is being studied by Inger[11] from a different approach. In his analysis the production term, or the source term, in the formal diffusion equation is approximated by a simpler function requiring only a correct matching near the surface. It is therefore based on a concept similar to that used herein in that the chemical state near the surface is assumed to be of controlling influence. It is still, however, an approximate analysis, and the accuracies of any of these approximate analyses can be checked only when exact solutions become available.

Conclusions

The concept of equivalent surface reaction has been developed for gas-phase recombination at the stagnation region of blunt bodies. This concept is based on the fact that the chemical state of a nonequilibrium highly cooled boundary layer is determined largely by the recombination that occurs near the wall. An equation based on this concept was shown to predict the heat transfer to a noncatalytic surface to within a few percent of the more accurate existing results. The equation was generalized to apply to the case in which surface catalytic recombination occurs simultaneously with gas-phase reaction. The present solution, which is in a simple closed form, should be useful in estimating heat transfer, although the accuracy of the solution cannot be checked for the general case, because of a lack of exact solutions. The atom concentration profile across the boundary layer can be obtained from the value m_w obtained herein, since in a frozen boundary layer the profile is determined when m_w is known (for instance, see Ref. 3), and the present theory is, in essence, based on the frozen boundary layer theory with modification only in the boundary condition at the surface. The profile thus obtained should approximate the true profile with the same degree of accuracy as the heat transfer results. This profile, in turn, defines the chemical state across the boundary layer. The equivalent surface reaction concept will be extended to flow fields beyond the stagnation point of a blunt body.

References

[1] Fay, J. A. and Riddell, F. R., "Theory of stagnation point heat transfer in dissociated air," J. Aerospace Sci. **25**, 73–85, 121 (1958).

[2] Chung, P. M. and Anderson, A. D., "Heat transfer around blunt bodies with nonequilibrium boundary layers," *Proceedings of the 1960 Heat Transfer and Fluid Mechanics Institute* (Stanford University Press, Stanford, Calif., 1960), pp. 150–161.

[3] Goulard, R. J., "On catalytic recombination rates in hypersonic stagnation heat transfer," Jet Propulsion **28**, 737–745 (1958).

[4] Chung, P. M. and Anderson, A. D., "Heat transfer to surfaces of finite catalytic activity in frozen dissociated hypersonic flow," NASA TN D-350 (1961).

[5] Chung, P. M., "Shielding stagnation surfaces of finite catalytic activity by air injection in hypersonic flight," NASA TN D-27 (1959).

[6] Scala, S. M., "Hypersonic stagnation point heat transfer to surfaces having finite catalytic efficiency," *Proceedings of the U. S. National Congress of Applied Mechanics* (Am. Soc. Mech. Engrs., New York, 1958), pp. 799–806.

[7] Inger, G. R., "Correlation of surface temperature effect on nonequilibrium heat transfer," ARS J. **32**, 1743–1744 (1962).

[8] Rae, W. J., "An approximate solution for the nonequilibrium boundary layer near the leading edge of a flat plate," Inst. Aerospace Sci. Paper 62-178 (June 1962).

[9] Goodwin, G. and Chung, P. M., "Effects of nonequilibrium flows on aerodynamic heating during entry into the earth's atmosphere from parabolic orbits," *Advances in Aeronautical Sciences* (Pergamon Press, New York, 1961), Vol. 4, pp. 997–1018.

[10] Chung, P. M., "A simplified study on the nonequilibrium Couette and boundary-layer flows with air injection," NASA TN D-306 (1960).

[11] Inger, G. R., "Nonequilibrium-dissociated stagnation point boundary layers with arbitrary surface catalycity," Aerospace Corp. Rept. ATN-63(9206)-3 (1963).

Subsonic Wing Span Efficiency

Richard C. Frost* and Robbie Rutherford†
General Dynamics/Fort Worth, Fort Worth, Tex.

THE rather arbitrary manner in which subsonic wing span efficiency often is selected has led to the development of an empirical method by which a more realistic and consistent value of efficiency can be determined. This method depends on a parameter R that is defined as the ratio of the actual chord force to the theoretical chord force with full leading-edge suction; R is found to be a function of leading-edge-radius Reynolds number. This note describes 1) the correlation of R with Reynolds number, and 2) the method for applying the results.

Figure 1 shows the results of plotting R vs Reynolds number with the characteristic length taken as the leading-edge radius at the wing mean aerodynamic chord. All of the data are for wings with symmetrical airfoils; R is determined as follows:

1) The drag at lift is given by

$$C_L{}^2/\pi \mathit{AR}e = C_c \cos\alpha + C_N \sin\alpha \tag{1}$$

With the usual assumption of small α and with $C_N = C_L/\cos\alpha$ Eq. (1) becomes

$$C_L{}^2/\pi \mathit{AR}e = C_c + C_L{}^2/C_{L\alpha} \tag{2}$$

Received January 2, 1963.

* Aerodynamics Group Engineer, Aeroscience Group.

† Aerodynamics Engineer, Aeroscience Group.